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UNSTEADY AXISYMMETRIC FLOWS IN THE APPROXIMATION OF SHALLOW WATER THEORY†

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A class of unsteady axisymmetric motions of an ideal incompressible fluid is investigated using exact solutions of the equations of the theory of shallow water.

In the axisymmetric case, the system of equations of the theory of shallow water in a cylindrical system of coordinates (z, r, φ) with a vertical z axis which coincides with the axis of symmetry, radial coordinate r and azimuthal angle φ, have the form [1]

$$\frac{du}{dt} - \frac{\upsilon^2}{r} = -g\frac{\partial H}{\partial r}, \quad \frac{d\upsilon}{dt} + \frac{u\upsilon}{r} = 0$$

$$\frac{\partial}{\partial t}(H-D) + \frac{1}{r}\frac{\partial}{\partial r}(ru(H-D)) = 0 \tag{1}$$

Here $u = u(r, t)$, $\upsilon = \upsilon(r, t)$ are the radial and azimuthal components of the velocity, $H = H(r, t)$ is the height of the free surface of the fluid, g is the acceleration due to gravity (directed downwards along the z axis) and $D(r)$ is a function describing the profile of the lower boundary (a diagram of the flow is shown in Fig. 1). The pressure p is determined from the hydrostatic relation: $p = p_0 + \sigma g(H-z)$, where σ is the fluid density and p_0 is the pressure on the upper boundary, which is assumed to be constant.

A solution is sought in the interval $r \in (0, r_*(t))$ in which $H \geqslant D$. The boundary $r = r_*(t)$ is defined by the equality $H(r_*(t), t) = D(r_*(t))$.

The initial condition will be fixed below.

We will confine ourselves to the case when the lower boundary has the form of a paraboloid of revolution

$$D(r) = \varkappa r^2/2 \tag{2}$$

We now rewrite system (1) in the form

$$\frac{du}{dt} = -g\frac{\partial h}{\partial r} - g\varkappa r + \frac{M^2}{r^3}, \quad \frac{dM}{dt} = 0$$

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(ruh) = 0, \quad h = H - \frac{1}{2}\varkappa r^2, \quad M = r\upsilon \tag{3}$$

We note that system (1) is invariant under a transition to a system of coordinates which rotates about the z axis at a constant angular velocity Ω. We know [2] that the dynamical equations are modified in the following way in such a system of coordinates: the terms $2\Omega\upsilon + \Omega^2 r$ are added to the right-hand side of the first equation of (1) and the term $2\Omega u$ is added to the right-hand side of the second equation, respectively. These terms are eliminated by making the substitution $\upsilon = \upsilon' - \Omega r$, and we arrive at the previous system (1) in which we now have υ' instead of υ.

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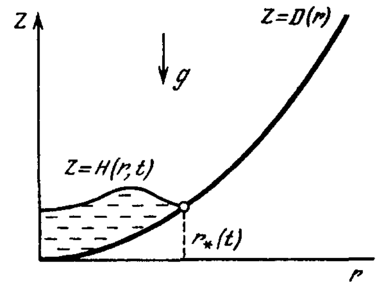

Fig. 1.

We now assume that, at the initial instant of time $t = 0$, the following are specified: the radial velocity in the form

$$u = Ar, \quad A = \text{const} \tag{4}$$

a certain arbitrary distribution of the angular moment $M = M_0(r)$, and the initial distribution of the thickness of the layer $h = h_0(r)$ linked with M_0 by the relation

$$\frac{M_0^2}{r^3} - g\frac{\partial h_0}{\partial r} = g\nu r, \quad \nu = \text{const.} \tag{5}$$

If $\varkappa = \text{const}$ and $\nu = \varkappa$, the given relation is compatible with the equality $u = 0$ and they describe the steady solution of system (3) (the cyclostrophic balance).

The problem is conveniently solved in Lagrangian coordinates. For $r = r(t, r_0)$, we seek a self-similar solution of the form

$$r = \rho(t)r_0 \tag{6}$$

where r_0 is the initial (Lagrangian) coordinate of the fluid particles. It follows from the second and third equations of (4) that

$$M = M_0(r_0), \quad h = \frac{r_0}{r}\frac{\partial r_0}{\partial r}h_0(r_0) = \frac{1}{\rho^2}h_0(r_0) \tag{7}$$

It can be seen from relation (6) and the second equality of (7) that $r^3\partial h/\partial r = r_0^3\partial h_0/\partial r_0$, and from this, when account is taken of the initial condition (5), we have

$$M^2 - gr^3\frac{\partial h}{\partial r} = g\nu r_0^4$$

On substituting this relation and expression (6) into the first equation of (3), after some reduction we obtain the final equation for the one-dimensional motion of a particle of unit mass with a potential energy $U = g\nu/2\rho^2 + g\varkappa\rho^2/2$

$$\frac{d^2\rho}{dt^2} = -\frac{\partial U}{\partial \rho}; \quad \rho = 1, \ \frac{d\rho}{dt} = A \text{ when } t = 0 \tag{8}$$

When $\varkappa = \text{const} > 0$ and $\nu > 0$, the motion consists of non-linear oscillations about the equilibrium position $\rho = (\nu/\varkappa)^{1/4}$. When $\nu \leqslant 0$, as will be seen from what follows, the solution $\rho(t)$ vanishes after a finite time.

When $\varkappa = \text{const} \geqslant 0$, Eq. (8) can be integrated in an elementary manner. The solution has the form

$$\rho(t) = \frac{1}{\sqrt{2g\varkappa}}\left(B + B'\sin(\varphi + \operatorname{sgn} A\sqrt{4g\varkappa}\,t)\right) \tag{9}$$

$$B = A^2 + g(\varkappa + \nu), \quad B' = \sqrt{B^2 - 4g\varkappa\nu}, \quad \sin\varphi = (2g\varkappa - B)/B'$$

If $\varkappa = 0$ (a plane lower boundary), we have

$$\rho(t) = \sqrt{(1 + At)^2 + g\nu t^2} \tag{10}$$

The transition to an Eulerian representation in this problem is simple on account of the simple dependence (6). We now write out a summary of the formulae which govern the final solution of the problem

$$u(r,t) = \frac{r}{\rho(t)}\frac{d\rho}{dt}, \quad M(r,t) = M_0\left(\frac{r}{\rho(t)}\right), \quad h = \frac{1}{\rho^2}h_0\left(\frac{r}{\rho(t)}\right) \tag{11}$$

where the function $\rho(t)$ is defined by relations (9) and (10) for the case when $\varkappa$ = const, while in the general case $\varkappa = \varkappa(t)$, it is determined by the solution of Eq. (8).

We will now consider some examples. The case when $\nu = \varkappa = 0$ (a plane lower boundary, cyclostrophic balance) is the simplest. It can be seen from the solution that an initial perturbation of the radial velocity leads to uniform motion along the radius while preserving the cyclostrophic balance. If $A < 0$, then, when $t \to |A|^{-1}$, we have $h \to \infty$ and $u \to \infty$, that is, the solution loses its meaning. In the case when $\varkappa = 0$, $\nu > 0$ and $A \geqslant 0$, the motion has the form of a monotonic spreading. When $A < 0$ and $t \in (0, |A|(A^2 + g\nu)^{-1})$ the fluid moves to the centre and, when $t > |A|(A^2 + g\nu)^{-1}$, it spreads monotonically. If $h_*(r_*) = 0$ for a certain r_*, the solution describes the spreading of a drop over a surface (ignoring surface tension). Finally, if $\varkappa$ = const > 0 and $\nu > 0$ (positive curvature of the lower boundary), the motion is of an oscillatory nature with a frequency $2\sqrt{(g\varkappa)}$.

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